Interfacial kinetic roughening with correlated noise

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Using an alternative scheme to generate correlated noise, we have reexamined issues of stochastic growth and directed polymer wandering subject to spatially correlated disorder. Our findings explicitly confirm relevant exponent equalities associated with the Kardar-Parisi-Zhang equation [Phys. Rev. Lett. **56**, 889 (1986)], thereby establishing important universal aspects. In addition, we show that the basin of attraction of the short-ranged fixed point function certainly extends to correlations falling inversely with separation, as previously conjectured.

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For various stochastic growth models [1], including Eden clusters, ballistic deposits, and restricted solid-on-solid (RSOS) algorithms, it is observed that the width $w(L,t)$ of the kinetically roughened surface evolves in accordance with the dynamic scaling form

$$w^2(L,t) \equiv \langle (h(x,t) - \langle h(x,t) \rangle)^2 \rangle \sim L^{2\chi} f(t/L^z) ,$$

where $h(x,t)$ is the interface height at position x and time t, the angular brackets denote the average over x in the system of size L at time t, and finally this quantity is averaged over the randomness. In addition, for $t \ll L^z, w(L,t) \sim t^\beta$ with $\beta = \chi/z$ and for $t \gg L^z$, $w(L,t) \sim L^\chi$. It is believed that a nonlinear stochastic Burgers equation, proposed by Kardar, Parisi, and Zhang [2] provides the best continuum description of these various growth models. Subsequently, Medina *et al.* [3] considered a generalization of the Kardar-Parisi-Zhang (KPZ) equation subject to correlated noise:

$$\partial_t h(x,t) = \nu \nabla^2 h(x,t) + \lambda/2 [\nabla h(x,t)]^2 + \eta(x,t) ,$$

where $\eta(x,t)$ represents the noise, with spatially correlated variance:

$$\langle \eta(x,t) \eta(x't') \rangle \sim |x - x'|^{2\rho - 1} \delta(t - t') .$$

Following Fourier transformation, the noise correlator becomes $\langle \eta(k,t) \eta(k',t') \rangle \sim k^{-2\rho} \delta(k+k') \delta(t-t')$. Because the noise has no temporal correlation, Galilean invariance is preserved and the exponents obey the characteristic KPZ scaling relation $\chi + z = 2$ [2–4], leaving only one independent scaling exponent. There have been several analytical attempts to pin down the value of the scaling exponents as a function of the correlation parameter ρ. From a one-loop perturbative dynamical renormalization-group (RG) analysis, Medina *et al.* [3] proposed

$$\beta(\rho) = \begin{cases} \frac{1}{3}, & 0 < \rho \le \frac{1}{4} \\ (1+2\rho)/(5-2\rho), & \frac{1}{4} < \rho \le 1 , \end{cases}$$

which follows, ultimately, from a nonrenormalization condition on the noise correlator. The same results were obtained quite independently by Halpin-Healy [5] via strong-coupling functional RG methods applied to the equivalent problem of directed polymers in random media (DPRM). Since the two methods are entirely complementary, the exactness of the above expression seems *highly likely.* Note that both these one-loop RG calculations yield $\rho_c = \frac{1}{4}$, below which the scaling is controlled by the white-noise fixed point, i.e., $\beta = \frac{1}{3}$ for all $\rho < \rho_c$. The recent KPZ analysis of Frey and Tauber [6] has revealed the nature of the short-ranged fixed point function to be unaltered by the inclusion of two-loop graphs, leaving robust the value $\rho_c = \frac{1}{4}$. In order to check this analytical prediction, two previous numerical studies [7,8] of the growth exponents have been made. Nevertheless, there remains a somewhat disconcerting spread among the quoted exponent values, as well as some important, unresolved issues needing clarification. This has motivated us to undertake our own numerical study of KPZ kinetic roughening subject to spatially correlated noise. Our work is of interest because we (i) use a precise prescription for the nontrivial task of generating appropriate spatial correlations, which we believe to be an improvement upon past methods [7,8]; (ii) explicitly illustrate for the reader (see Fig. 1) the quality of our correlated noise; (iii) make a point of confirming the characteristic KPZ exponent quality directly (see Fig. 4), thereby affirming the broader notion of KPZ universality; and, finally (iv) support the qualitative prediction of the one-loop dynamic RG (see Fig. 3) while providing strong evidence that the early-time exponent sticks to its uncorrelated KPZ value $\frac{1}{3}$ for correlations falling faster than $\rho \le 0$. This verifies commonly preached [5,9] but hitherto unproved dogma citing dimensional considerations for δ-function correlated noise.

We note that previous simulations [7,8] generated noise directly via the relation $\langle \eta(k,t) \eta(k',t') \rangle \sim k^{-2\rho} \delta(k + k') \delta(t - t')$, which is obtained in the continuum limit, i.e., infinitely large system size. However, for a discrete lattice of finite size, the above k-space relation can only generate the noise with the advertised power-law correlation over a very small range in real space. Typically, to maintain the desired spatial correlation within the size of the system studied, it was necessary to transform 10^3 times more numbers than the lattice size. Here we propose an alternative algorithm to generate spatially correlated noise. By our algorithm, the noise obeys perfectly well the power-law correlation for the whole range of the

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lattice. In addition, it respects the periodic boundary conditions, which is essential in order to compare the simulation of the lattice to the continuum limit of the KPZ equation. For our noise construction algorithm on a discrete lattice, $\eta(x,t)$ has the spatial correlation

$$\langle \eta(x,t)\eta(x',t')\rangle = g_\rho(x-x')\delta_{t,t'} ,$$

where

$$g_\rho(x-x') = \begin{cases} g_\rho(0), & x=x' \\ |x-x'|^{2\rho-1}, & |x-x'|=1,2,3,\ldots . \end{cases}$$

The value of $g_\rho(0)$ must be assigned in this discrete version, the natural choice being

$$g_\rho(0) \equiv 2\int_0^{1/2} du\, g_\rho(u) = 2\int_0^{1/2} du\, u^{2\rho-1} .$$

After performing the discretized Fourier transformation on a lattice of size $2N$, we get

$$\langle \eta(k,t)\eta(k',t')\rangle = \delta_{k+k',0}\delta_{t,t'}S_\rho(k)$$

with

$$S_\rho(k) = \frac{1}{2N}\sum_{u=-N}^{N-1} g_\rho(u)e^{-iku} .$$

We can now generate noise satisfying the above relation in k space as

$$\eta(k,t) \equiv \sqrt{S_\rho(k)}(r_k - \tfrac{1}{2})\exp(2\pi i \phi_k)$$

and

$$\eta(-k,t) \equiv \eta^*(k,t) ,$$

where r_k, ϕ_k are independent uniform random variables between 0 and 1 and $k=(2\pi/2N)(-N)$, $(2\pi/2N)(-N+1),\ldots,(2\pi/2N)(N-1)$. After Fourier transformation back to real space, we obtain the noise $\eta(x,t)$ of the desired spatial correlation. In Fig. 1 we show the noise correlator as a function of spatial separation, compared with the expected $r^{2\rho-1}$ behavior. The high quality of our spatially correlated noise is evident.

We next use this construction in a numerical study of DPRM [10] and RSOS [11] models subject to spatially correlated noise. For uncorrelated white noise, these two models are exemplary members of the KPZ universality class, with well-documented, very reliable scaling properties [1]. We employ the discretized DPRM on a square lattice with the transverse direction labeled x and the longitudinal direction labeled t. At zero temperature, the recursion relation for the minimal energy path reads

$$E(x,t) = \min[E(x,t-1)+\eta(x,t-1), E(x-1,t-1) + \eta(x-1,t-1)+\gamma, E(x+1,t-1) + \eta(x+1,t-1)+\gamma] ,$$

where γ denotes the DPRM bending energy and $\eta(x,t)$ is the random site energy [10]. The transverse fluctuation is defined as $\Delta x(t)^2 \equiv \langle [x-x_0(x,t)]^2\rangle$, where $x_0(x,t)$ is the origin of the optimal walk termining at each (x,t). The angular brackets denote the average over all positions x in the system of size L at time t. Furthermore, this quantity is averaged over many different realizations of the random site energies. $\Delta x(t)$ is expected to scale as t^ζ, when $1 \ll t^\zeta \ll L$. Correspondingly, the energy fluctuation $\Delta E(t)^2 \equiv \langle (E(x,t) - \langle E\rangle)^2\rangle$ scales as $t^{2\omega}$, when $1 \ll t^\zeta \ll L$. It is well known [1] that the directed polymer problem can be mapped onto the KPZ equation by a simple transformation, with the DPRM and kinetic roughening exponents related via $\zeta = 1/z$ and $\omega=\beta$, respectively. Consequently, within the DPRM context, the characteristic KPZ exponent equality reads $\omega = 2\zeta - 1$.

Our simulation was performed on a very large system lattice $L = 65\,536 = 2^{16}$, with the random site energies generated from our noise construction algorithm, a bending energy $\gamma = 0.5$, introduced in [10], checked by us, and statistical averaging performed over 100 runs. The DPRM exponents ζ and ω were determined by the standard model, using double logarithmic plots of the transverse (and energy) fluctuations versus time. Note that we have determined ζ and ω *independently* in order to explicitly check the KPZ scaling relation in the presence of spatially correlated noise.

We first did the simulation with the uncorrelated site potential to check Kim's choice [10] for the bending energy γ and obtained $\zeta = 0.633 \pm 0.005$ and $\omega = 0.333 \pm 0.003$, in excellent agreement with the exact values $\frac{2}{3}$ and $\frac{1}{3}$, respectively. Figure 2 shows our DPRM (and RSOS; see later) results for spatial correlation parameter $\rho = \frac{1}{4}$. As appreciated previously [8], this case is crucial from the point of view of the one-loop dynamic RG calculation. Our numerical results indicate $\zeta = 0.692 \pm 0.005$ and $\omega = 0.364 \pm 0.005$ for $\rho = \frac{1}{4}$, the former being consistent with a very robust, earlier finding [8] of $\zeta = 0.688$. We then varied the spatial correlation of the noise, collecting our results in Fig. 3, which shows our measured $\omega_{\rm DPRM}$ as a function of the noise correlation parameter ρ along with the theoretical prediction of the renormalization-group treatments. In addition, we have extended the simulation to the regime of the noise correlation exponent $\rho < 0$, with $\rho = -0.2, -0.4, -0.5, -1, -1.5$, and the exponents ζ and ω do indeed stick to the exact values 2/3 and 1/3

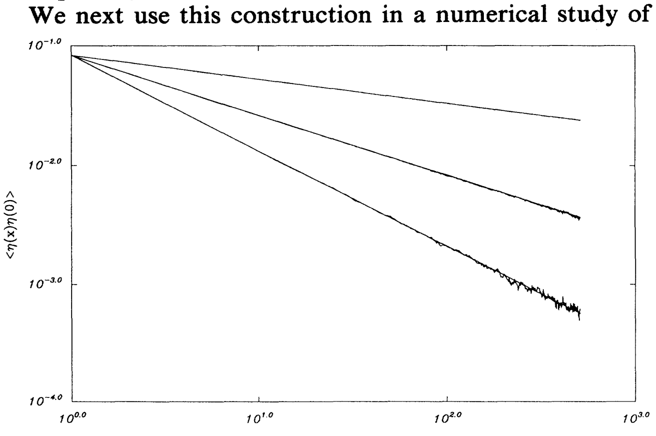

FIG. 1. Variance of our spatially correlated noise as compared with the expected behavior: $\langle \eta(x)\eta(0)\rangle \sim x^{2\rho-1}$. The curves, top to bottom, correspond to $\rho = 0.4, 0.25, 0.1$, respectively.

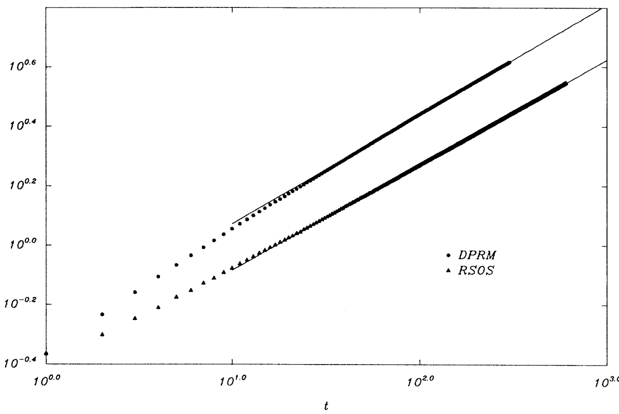

FIG. 2. Double log plot of the energy (height) fluctuations of the DPRM (RSOS) models in the early-time regime of these two correlated KPZ models. The straight lines indicate $\omega_{\mathrm{DPRM}} = 0.364 \pm 0.005$ (upper) and $\beta_{\mathrm{RSOS}} = 0.353 \pm 0.005$ (lower), respectively, for the value $\rho = \frac{1}{4}$ of the spatial correlation parameter.

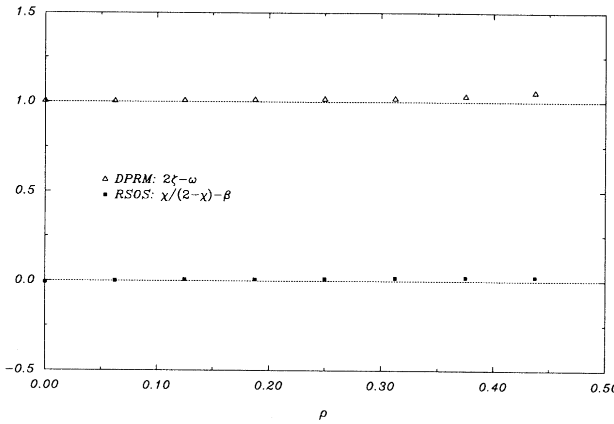

FIG. 4. Test of the KPZ exponent equality for correlated DPRM (△) and RSOS (■) models. For the former, we expect $\omega = 2\zeta - 1$, while the latter should show $\beta = \chi / z$, where $\chi + z = 2$. It is apparent that the fundamental KPZ relation is quite robust in the presence of spatially correlated noise. Interestingly, it is well preserved despite the correction to scaling phenomena associated with $\rho_c = \frac{1}{4}$.

known rigorously for spatially uncorrelated noise. This had been asserted previously [5,9] and is numerically confirmed here. In Fig. 4 we show that our results for the exponents ζ and ω obey the KPZ scaling relation $2\zeta - \omega = 1$ excellently for all values of ρ. The largest deviation is about 5%. We notice that the deviation increases for higher ρ, a consequence of finite size effects.

Next we study the RSOS [11] model subject to spatially correlated noise. We generate a representation of spatially correlated noise, which gives the growth probability at each site x. That is, if the noise $\eta(x,t) > 0$, then $h(x,t) = h(x,t-1) + 1$ subject to the restriction $|h(x,t) - h(x \pm 1, t)| \leq 1$; the growth is rejected if $\eta(x,t) < 0$. In addition, a parallel growth algorithm is used in which

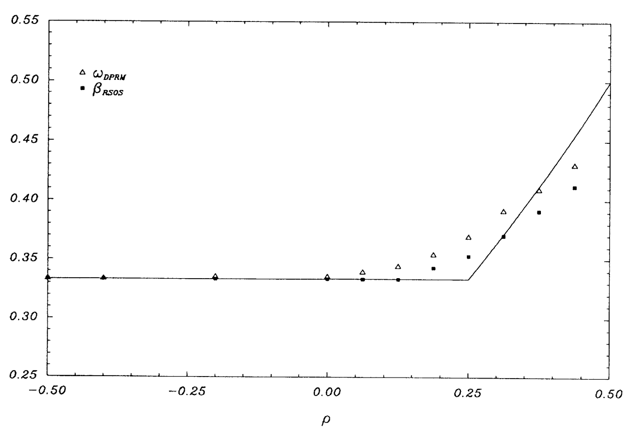

FIG. 3. KPZ exponents ω_{DPRM} (△) and β_{RSOS} (■) as a function of the noise correlation parameter ρ, along the theoretical prediction (solid line) following from one-loop KPZ dynamic RG [3] and DPRM functional RG [5] calculations. Our numerics show convincingly that the exponents stick to their uncorrelated value $\frac{1}{3}$ for all $\rho < 0$. For positive ρ, corrections to scaling initiate a clear deviation from the exact RG prediction.

growth is attempted on alternate odd (and even) sublattices on odd (and even) time steps. After each pair of sublattices is updated, a noise distribution $\eta(x,t)$ is generated. According to the growth rule of the RSOS model, the correlation of the growth in different sites x, x' at the same time slice depends on $P(\eta(x,t)\eta(x',t) > 0)$ instead of $\langle \eta(x,t)\eta(x',t) \rangle$. We have explicitly verified that, by our way of our noise construction, $P(\eta(x,t)\eta(x',t) > 0)$ also has this corresponding spatial correlation, i.e.,

$$P(\eta(x,t)\eta(x',t) > 0) - 0.5 \sim |x - x'|^{2\rho - 1}$$

when

$$\langle \eta(x,t)\eta(x',t) \rangle \sim |x - x'|^{2\rho - 1} .$$

Our RSOS simulation was performed on a system of size $L = 65\,536 = 2^{16}$ with 600 time steps, averaged over 100 runs. As a check of the growth algorithm, we first performed the simulation with the uncorrelated noise and obtained the early time exponent $\beta = 0.330 \pm 0.003$, once again in excellent agreement with the exact value $\frac{1}{3}$. We then applied the simulation with the correlated noise for many values of ρ. Our findings for $\rho = \frac{1}{4}$ spatially correlated RSOS in the early-time regime are shown alongside the DPRM results in Fig. 2, strong testimony to correlated KPZ universality. We have, for this value of the spatial correlation parameter, $\beta_{\mathrm{RSOS}} = 0.353 \pm 0.005$. Results for this early-time exponent as a function of the noise correlation exponent ρ are also shown in Fig. 3. Note that our RSOS values agree best with the prediction of Medina *et al.* [3] for small positive ρ, where it is appears that the true white-noise fixed-point function maintains some influence. For spatial correlations larger than $\rho \approx \frac{1}{8}$, however, our RSOS data clearly begin to deviate from the one-loop dynamical renormalization-group prediction. This can be discerned in previous work [7,8] as

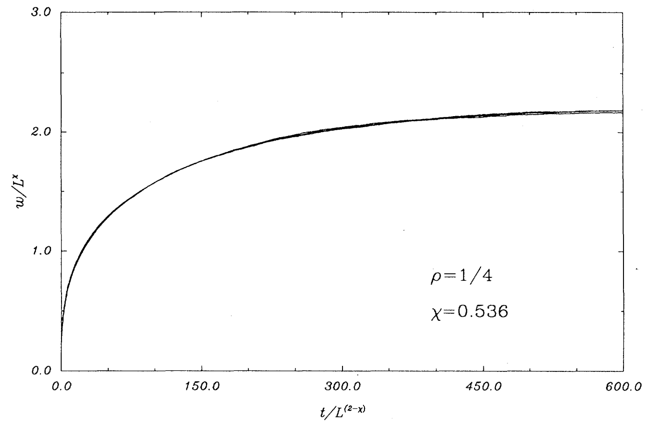

FIG. 5. Test of the dynamic scaling hypothesis for the RSOS model subject to spatially correlated noise, with $\rho=\frac{1}{4}$. We obtain excellent data collapse for system sizes $L=64,128,256$, assuming $\chi=0.536\pm0.005$.

well. By comparison, the DPRM data break off even earlier.

In addition to the direct measurement of the early-time exponent β, we have determined χ independently, via the data collapse of $w(L,t)$, in order to check the scaling relation $\beta=\chi/z$. We simulate the growth with system size $L=64,128,256$ in the time duration $t=1000,2000,5000$, respectively, averaged over 10 000 runs. Then, by varying the value of χ, under the restriction $\chi+z=2$, we get the excellent data collapse, affirmation of the dynamic scaling hypothesis for KPZ subject to correlated noise. Figure 5 illustrates our findings for $\rho=\frac{1}{4}$. Note that, in the absence of the KPZ nonlinearity $\lambda=0$, it is possible to calculate analytically the full scaling function in closed form, including the dependence on the spatial correlation parameter ρ [12]. In Fig. 4 we show that our results for the RSOS growth exponents, which reveal that they obey the relation $\beta=\chi/z$, with $z+\chi=2$, rather well for all values of ρ, prove that spatially correlated RSOS still lies within the realm of KPZ. We note that the discrepancy here for RSOS, as for the DPRM, increases with the ρ value, although somewhat more modestly.

In conclusion, we propose our own noise construction algorithm that generates correlations obeying the desired power laws throughout the entire system, while respecting imposed periodic boundary conditions. Extensive numerical studies of DPRM and RSOS growth models subject to spatially correlated noise have been undertaken. We have measured the growth exponents independently, i.e., ζ and ω for DPRM, χ and β in RSOS, as functions of the noise correlation parameter ρ. Interestingly, we also simulate the regime $\rho<0$. As a critical feature, we have explicitly verified the characteristic KPZ exponent relation in the presence of spatially correlated noise, confirming broader utility of the KPZ equation. Moreover, for $\rho<0$, we provide strong evidence that scaling indices most certainly stick to values directed by the white-noise fixed-point function [3], supporting earlier suspicions [5,9]. Finally, as in past numerical efforts on spatially correlated RSOS [7] and DPRM [8] models, we find modest disagreements with the one-loop dynamic and functional RG predictions as $\rho\rightarrow\frac{1}{2}$. Indeed, since the short-ranged fixed point function loses stability at $\rho_c=\frac{1}{4}$, where an irrelevant operator becomes marginal, it is perhaps likely that numerically measured exponents are effective and not truly asymptotic. Indeed, allowing for a logarithmic correction to the anticipated power law at $\rho_c=\frac{1}{4}$ leads to a fit of comparable quality as in Fig. 4. Work is presently in progress to understand more generally correction to scaling phenomena in this context.

Note added in proof: Following acceptance of this publication, we became aware of related independent work of Makse *et al.* [13] for generating high quality correlated noise.

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